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# On a Theorem of Hille, Szegö, and Tamarkin

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1. In answer to a question raised by the chemist Mendeleieff, A. Markoff [2] proved the following theorem.

THEOREM A. *If* $P_n(x) = \sum_{\nu=0}^{n} c_\nu x^\nu$ *is a polynomial of degree* $n$ *and* $|P_n(x)| \leqslant 1$ *in the interval* $-1 \leqslant x \leqslant 1$, *then in the same interval*

$$|P_n'(x)| \leqslant n^2. \tag{1.1}$$

The constant $n^2$, in (1.1), cannot be replaced by any smaller constant, in fact, there is a well-known polynomial which satisfies the conditions of Theorem A and whose derivative at the point $x = 1$ is equal to $n^2$. This polynomial is the $n$-th Tchebycheff polynomial,

$$T_n(x) = \cos(n \arccos x) = 2^{n-1} \prod_1^n \left\{x - \cos\left(\frac{(\nu - \frac{1}{2})\pi}{n}\right)\right\}. \tag{1.2}$$

Einar Hille, G. Szego, and J. D. Tamarkin [1] extended Markoff's theorem by proving the following

THEOREM B. *Let* $p > 1$ *and let* $P_n(x)$ *be an arbitrary rational polynomial of degree* $n$. *Then*

$$\left\{\int_{-1}^{1} |P_n'(x)|^p \, dx\right\}^{1/p} \leqslant An^2 \left\{\int_{-1}^{1} |P_n(x)|^p \, dx\right\}^{1/p}, \tag{1.3}$$

*where* $A$ *is a constant which depends only on* $p$, *but not on* $f(x)$ *or on* $n$.

The factor $n^2$ in (1.3) cannot be replaced by any function tending to infinity more slowly. In fact, for each $n$ there exist polynomials $P_n(x)$ of degree $n$, such that the left member of (1.3) is $\geqslant Bn^2$, where $B$ is a constant of the same nature as $A$.

However, we observe that only a little restriction on the location of the zeros of $P_n(z)$ allows us to replace $n^2$ by $n^{1+1/p}$. In fact, we have the following

THEOREM. *Let $p > 1$ and let $P_n(z)$ be a real-valued rational polynomial of degree n. Let $P_n(z)$ have no zeros in the two circular regions*

$$| z \pm a | < 1 - a \qquad (0 \leqslant a < 1); \tag{1.4}$$

*then*

$$\left\{\int_{-1}^{1} | P_n'(x)|^p \, dx\right\}^{1/p} \leqslant Bn^{1+1/p} \left\{\int_{-1}^{1} | P_n(x)|^p \, dx\right\}^{1/p},$$

*where B is a constant which depends only on p and a, but not on $P_n(x)$ or on n.*

Note that $a$ can be taken as close to 1 as we like, except that $1 - a$ has to be positive. Thus, we have the interesting conclusion that

$$\left\{\int_{-1}^{1} | P_n'(x)|^p \, dx\right\}^{1/p} : \left\{\int_{-1}^{1} | P_n(x)|^p \, dx\right\}^{1/p} = O(n^{1+1/p}), \tag{1.5}$$

howsoever small the two exceptional circles of the theorem may be.

2. For the proof of the Theorem, we shall need the following two lemmas, one of which is due to Gabriel and the other is an extension of a classical theorem of S. Bernstein-M. Riesz (Hille, Szegö, and Tamarkin, see Lemmas 2.1 and 2.2).

LEMMA 2A. *If $\Gamma$ is any convex closed curve in the complex z plane, and C any convex curve inside $\Gamma$, and if $F(z)$ is regular inside and on $\Gamma$, then*

$$\int_C |F(z)|^\lambda \, | dz | \leqslant G \int_\Gamma |F(z)|^\lambda \, | dz | . \tag{2.1}$$

*Here, $\lambda$ is any number $> 0$, and G an absolute constant.*

Let $C$ be any simple rectifiable Jordan curve in the complex $z$ plane and let

$$z = \psi(w) = cw + c_0 + c_1 w^{-1} + \cdots + c_n w^{-n} \cdots , \qquad c > 0,$$

be the function which maps the simply connected infinite domain exterior to $C$ conformally onto the exterior of the unit circle $| w | = 1$ in the $w$ plane. Let $C_R$ be the image in the $z$ plane of the circle $| w | = R$. With this notation we have (Hille, Szegö, and Tamarkin, Lemma 2.2.).

LEMMA 2B. *If $P_n(z)$ is any polynomial of degree $n$, then*

$$\int_{C_R} | P_n(z)|^p \, | dz | \leqslant R^{np+1} \int_C | P_n(z)|^p \, | dz | , \qquad p > 0. \tag{2.2}$$

3. *Proof of Theorem.* By Cauchy's formula

$$P_n'(1) = \frac{1}{2\pi i} \int_{|w-1|=1/n} \frac{P_n(w)}{(w-1)^2} \, dw,$$

so that

$$| P_n'(1)| \leqslant \frac{n}{2\pi} \int_0^{2\pi} \left| P_n\left(1 + \frac{1}{n} e^{i\theta}\right) \right| d\theta. \tag{3.1}$$

We shall prove that if $P_n(z)$ is real-valued and does not vanish in the two circular regions defined by (1.4), then for $0 \leqslant \theta < 2\pi$

$$\begin{aligned} \left| P_n\left(1 + \frac{1}{n} e^{i\theta}\right) \right| &\leqslant \left\{1 + \frac{1 + \cos\theta}{n(1-a) - 1}\right\}^{n/2} \left| P_n\left(1 - \frac{1}{n}\right) \right| \\ &< \left| P_n\left(1 - \frac{1}{n}\right) \right| \exp\left(\frac{1}{1-a}\right). \end{aligned} \tag{3.2}$$

Thus, we can conclude that

$$| P_n'(1)| < n \left| P_n\left(1 - \frac{1}{n}\right) \right| \exp\left(\frac{1}{1-a}\right). \tag{3.3}$$

Now let $(1 + a)/2 \leqslant b \leqslant 1$ and let us consider the polynomial $P_n(bx)$, which has the following properties. It is real-valued, its degree is $n$, and it does not vanish in the circles $| z \pm a/b | < (1 - a)/b$, therefore, certainly not in the discs $| z \pm a/b | < 1 - a/b$. Applying (3.3) to the polynomial $P_n(bx)$, one finds

$$\begin{aligned} b \, | P_n'(b)| &\leqslant n \left| P_n\left(b - \frac{b}{n}\right) \right| \exp\left\{\left(1 - \frac{a}{b}\right)^{-1}\right\} \\ &\leqslant n \left| P_n\left(b - \frac{b}{n}\right) \right| \exp\left(\frac{1+a}{1-a}\right), \end{aligned}$$

whence

$$| P_n'(b)| \leqslant \frac{2n}{1+a} \left| P_n\left(b - \frac{b}{n}\right) \right| \exp\left(\frac{1+a}{1-a}\right) \qquad \text{for} \qquad \frac{1+a}{2} \leqslant b \leqslant 1. \tag{3.4}$$

Hence,

$$\int_{(1+a)/2}^{1} | P_n'(x)|^p \, dx < \left(\frac{2}{1+a}\right)^p n^p \exp\left\{\frac{p(1+a)}{1-a}\right\} \int_{(1+a)/2}^{1} \left| P_n\left(x - \frac{x}{n}\right)\right|^p dx$$
$$< \left(\frac{2}{1+a}\right)^p n^p \exp\left\{\frac{p(1+a)}{1-a}\right\} \int_{0}^{1} | P_n(x)|^p \, dx.$$

Similarly, we can prove that

$$\int_{-1}^{-(1+a)/2} | P_n'(x)|^p \, dx < \left(\frac{2}{1+a}\right)^p n^p \exp\left\{\frac{p(1+a)}{1-a}\right\} \int_{-1}^{0} | P_n(x)|^p \, dx.$$

Thus,

$$\int_{1 \geqslant |x| > (1+a)/2} | P_n'(x)|^p \, dx < \left(\frac{2}{1+a}\right)^p n^p \exp\left\{\frac{p(1+a)}{1-a}\right\} \int_{-1}^{1} | P_n(x)|^p \, dx. \tag{3.5}$$

This last inequality depends on the validity of the estimate (3.2) which we now proceed to prove.

If $z_\nu = x_\nu + iy_\nu$ is a complex root of $P_n(z)$, then $(z - z_\nu)(z - \bar{z}_\nu)$ is a factor of $P_n(z)$, and for $0 \leqslant \theta < 2\pi$ one has

$$\left| \frac{\left(1 + \frac{1}{n} e^{i\theta} - z_\nu\right)\left(1 + \frac{1}{n} e^{i\theta} - \bar{z}_\nu\right)}{\left(1 - \frac{1}{n} - z_\nu\right)\left(1 - \frac{1}{n} - \bar{z}_\nu\right)} \right|$$

$$= \frac{\left\{\left[x_\nu^2 + \left(1 + \frac{1}{n}\cos\theta\right)^2 - 2x_\nu\left(1 + \frac{1}{n}\cos\theta\right) + y_\nu^2 + \frac{1}{n^2}\sin^2\theta\right]^2 - 4\frac{1}{n^2} y_\nu^2 \sin^2\theta\right\}^{1/2}}{x_\nu^2 + y_\nu^2 + \left(1 - \frac{1}{n}\right)^2 - 2x_\nu\left(1 - \frac{1}{n}\right)}$$

$$\leqslant \frac{x_\nu^2 + \left(1 + \frac{1}{n}\cos\theta\right)^2 - 2x_\nu\left(1 + \frac{1}{n}\cos\theta\right) + y_\nu^2 + \frac{1}{n^2}\sin^2\theta}{x_\nu^2 + y_\nu^2 + \left(1 - \frac{1}{n}\right)^2 - 2x_\nu\left(1 - \frac{1}{n}\right)} \tag{3.6}$$

$$= 1 + \frac{2}{n} \frac{(1 - x_\nu)(1 + \cos\theta)}{x_\nu^2 + y_\nu^2 + \left(1 - \frac{1}{n}\right)^2 - 2x_\nu\left(1 - \frac{1}{n}\right)}$$

$$\leqslant 1 + \frac{1 + \cos\theta}{n(1 - a) - 1},$$

where $n(1-a)-1 \neq 0$ if $n$ is sufficiently large. The last inequality is justified by the following:

We have to show that

$$\frac{1-x_\nu}{x_\nu^2+y_\nu^2+\left(1-\frac{1}{n}\right)^2-2x_\nu\left(1-\frac{1}{n}\right)} \leqslant \frac{n}{2\{n(1-a)-1\}},$$

which is equivalent to

$$x_\nu^2-2ax_\nu+a^2+y_\nu^2 \geqslant (1-a)^2-\frac{1}{n^2},$$

which is certainly true since

$$|z-a|^2>(1-a)^2>(1-a)^2-\frac{1}{n^2},$$

by hypothesis.

In case the root $z_\nu = x_\nu$ is real, we have directly

$$\left|\frac{1+\frac{1}{n}e^{i\theta}-x_\nu}{1-\frac{1}{n}-x_\nu}\right| = \left\{1+\frac{2}{n}\,\frac{(1-x_\nu)(1+\cos\theta)}{x_\nu^2-2x_\nu\left(1-\frac{1}{n}\right)+\left(1-\frac{1}{n}\right)^2}\right\}^{1/2}$$

$$\leqslant \left\{1+\frac{1+\cos\theta}{n(1-a)-1}\right\}^{1/2}. \tag{3.7}$$

The inequality (3.2) follows from (3.6) and (3.7).

We are now going to prove that for every $p>1$

$$\left\{\int_{|x|\leqslant(1+a)/2} |P_n'(x)|^p\,dx\right\}^{1/p} < Cn^{1+1/p}\left\{\int_{-1}^{1}|P_n(x)|^p\,dx\right\}^{1/p}, \tag{3.8}$$

where $C$ is a constant which depends only on $p$ and $a$, but not on $P_n(x)$ or on $n$.

Let $n$ be so large that

$$\frac{1+a}{2} \leqslant \left[\frac{1}{2}\left(1+\frac{1}{n}+\frac{1}{1+\frac{1}{n}}\right)\right]^{-1}, \tag{3.9}$$

and let $P_n(x)$ be an arbitrary polynomial of degree $n$. By Cauchy's formula we have

$$P_n'(x) = \frac{1}{2\pi i}\int_{C_x}\frac{P_n(z)}{(z-x)^2}\,dz;$$

here $-(1+a)/2 \leqslant x \leqslant (1+a)/2$ and the contour of integration $C_x$ will be specified later. Hölder's inequality yields

$$\int_{-(1+a)/2}^{(1+a)/2} |P_n'(x)|^p \, dx$$

$$= (2\pi)^{-p} \int_{-(1+a)/2}^{(1+a)/2} dx \left| \int_{C_x} \frac{P_n(z)}{(z-x)^2} dz \right|^p$$

$$\leqslant (2\pi)^{-p} \int_{-(1+a)/2}^{(1+a)/2} dx \left\{ \int_{C_x} |P_n(z)|^p \, |dz| \right\} \left\{ \int_{C_x} \frac{|dz|}{|z-x|^{2p'}} \right\}^{p/p'}, \qquad (3.10)$$

where $1/p + 1/p' = 1$.

Let $R = 1 + 1/n$ and let $E_R$ denote the ellipse with foci at $\pm 1$ and semi-axes $\frac{1}{2}(R + R^{-1})$ and $\frac{1}{2}(R - R^{-1})$. It is easy to verify that for $|x| \leqslant (1+a)/2 \leqslant 2(R + R^{-1})^{-1}$ the shortest distance $D = D(x, R)$ of $x$ from $E_R$ is $\frac{1}{2}(R - R^{-1})(1 - x^2)^{1/2}$. Now, choose for $C_x$, the circle with the centre $x$ and radius $D$; this circle is internally tangent to $E_R$. Then the last factor in the right member of (3.10) does not exceed

$$(2\pi \, D^{1-2p'})^{p/p'} = (2\pi)^{p/p'} \, D^{-p-1}.$$

As for the first expression in braces in (3.10), a successive application of Lemmas 2A and 2B yields

$$\int_{C_x} |P_n(z)|^p \, |dz| \leqslant G \int_{E_R} |P_n(z)|^p \, |dz| \leqslant 2GR^{np+1} \int_{-1}^{1} |P_n(x)|^p \, dx.$$

On substituting these results in (3.10), we have

$$\int_{-(1+a)/2}^{(1+a)/2} |P_n'(x)|^p \, dx \leqslant \pi^{-1} G R^{np+1} \int_{-1}^{1} |P_n(x)|^p \, dx \int_{-(1+a)/2}^{(1+a)/2} \{D(x, R)\}^{-p-1} \, dx.$$

But

$$\begin{aligned}
\int_{-(1+a)/2}^{(1+a)/2} \{D(x, R)\}^{-p-1} \, dx &= 2 \left( \frac{R - R^{-1}}{2} \right)^{-p-1} \int_0^{(1+a)/2} (1 - x^2)^{-(p+1)/2} \, dx \\
&\leqslant 2 \left( \frac{R - R^{-1}}{2} \right)^{-p-1} \int_0^{(1+a)/2} (1 - x)^{-(p+1)/2} \, dx \\
&= 2 \left( \frac{R - R^{-1}}{2} \right)^{-p-1} \frac{2}{p-1} \left\{ \left( \frac{1-a}{2} \right)^{-(p-1)/2} - 1 \right\} \\
&< \frac{4}{p-1} \, n^{p+1} \left\{ \left( \frac{1-a}{2} \right)^{-(p-1)/2} - 1 \right\}.
\end{aligned}$$

Hence, if $n$ is so large that (3.9) holds, then

$$\int_{-(1+a)/2}^{(1+a)/2} | P_n'(x)|^p \, dx$$

$$< 4\pi^{-1}Ge^p(p-1)^{-1} \left\{ \left( \frac{1-a}{2} \right)^{-(p-1)/2} - 1 \right\} n^{p+1} \int_{-1}^{1} | P_n(x)|^p \, dx.$$

From this, (3.8) follows, since (1.3) holds for every $n$.

Combining (3.5) and (3.8) we get the Theorem.

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## References

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